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Probabilités/*Probability Theory*
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A Bennett Concentration Inequality and Its Application to Suprema of Empirical Processes

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Abstract. We introduce new concentration inequalities for functions on product spaces They allow to obtain a Bennett type deviation bound for suprema of empirical processes indexed by upper bounded functions. The result is an improvement on Rio's version [6] of Talagrand's inequality [7] for equidistributed variables. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Une inégalité de concentration de type Bennett et son application aux maxima de processus empiriques

Résumé. *Nous proposons deux inégalités de concentration pour des fonctions de n variables indépendantes. L'une d'elles permet d'obtenir une inégalité de déviation de type Bennett pour les processus empiriques indexés par des classes de fonctions bornées à droite. Cela améliore la version donnée par Rio [6] de l'inégalité de Talagrand [7] pour des observations équidistribuées.* © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

1. Introduction

Soient $X_1, \ldots, X_n$ n variables indépendantes de loi jointe P à valeurs dans un espace polonais $\mathcal{X}$. Soit F une fonction mesurable de $\mathcal{X}^n$ dans $\mathbb{R}$. Nous étudions la concentration de la variable aléatoire $Z := F(X_1, \ldots, X_n)$ par rapport à sa moyenne. Nous donnons des conditions sur Z qui permettent d'obtenir une inégalité exponentielle de type Bennett ainsi que des conditions plus générales qui permettent d'obtenir un contrôle de la transformée de Laplace de type Bernstein. Nous montrons ensuite que ces résultats s'appliquent aux maxima de processus empiriques, c'est-à-dire aux variables aléatoires $Z = \sup_{f\in\mathcal{F}} \sum_{i=1}^n f(X_i)$ telles que $\mathcal{F}$ soit un ensemble de fonctions mesurables de $\mathcal{X}$ dans $\mathbb{R}$ de carré intégrable sous P.

$\mathcal{A}$ est la σ-algèbre engendrée par $(X_1, \ldots, X_n)$ et pour tout $k \in \{1, \ldots, n\}$, $\mathcal{A}_n^k$ est la σ-algèbre engendrée par $(X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n)$. $\mathbb{E}_n^k\,[\cdot]$ est l'éspérance par rapport à $\mathcal{A}_n^k$.

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2. Résultats Principaux

Le premier résultat donne une inégalité de type Bennett qui peut être considérée comme une généralisation d'un théorème de Boucheron, Lugosi et Massart [1].

THEOREM 2.1. – *Soient $(Z, Z'_1, \ldots, Z'_n)$ une séquence de v.a. $\mathcal{A}$-mesurables et $(Z_k)_{k=1,\ldots,n}$ une séquence de v.a. respectivement $\mathcal{A}_n^k$-mesurables. Supposons qu'il existe un réel positif u tel que, pour tout $k = 1, \ldots, n$ les inégalités suivantes sont vraies*

$$Z'_k \leqslant Z - Z_k \leqslant 1 \ p.s.\,, \quad \mathbb{E}_n^k\left[Z'_k\right] \geqslant 0 \ \ et \ \ Z'_k \leqslant u \ p.s. \tag{1}$$

Soit alors σ un réel tel que $\sigma^2 \geqslant \frac{1}{n}\sum_{k=1}^n \mathbb{E}_n^k\left[(Z'_k)^2\right]$ presque sûrement. Posons $v = (1+u)\mathbb{E}[Z] + n\sigma^2$, $\psi(\lambda) = \exp(-\lambda) - 1 + \lambda$ et $h(x) = (1+x)\log(1+x) - x$. Si la condition suivante est vérifiée

$$\sum_{k=1}^n Z - Z_k \leqslant Z \ p.s.\,,$$

nous obtenons pour tout $\lambda \geqslant 0$,

$$\log \mathbb{E}\left[e^{\lambda(Z - \mathbb{E}[Z])}\right] \leqslant \psi(-\lambda)v\,,$$

ce qui donne, pour tout $x \geqslant 0$

$$\mathbb{P}\left[Z \geqslant \mathbb{E}[Z] + x\right] \leqslant \exp\left(-vh\left(\frac{x}{v}\right)\right)\,.$$

et

$$\mathbb{P}\left[Z \geqslant \mathbb{E}[Z] + \sqrt{2vx} + \frac{x}{3}\right] \leqslant e^{-x}\,.$$

Le second résultat donne un contrôle de type Bernstein de la transformée de Laplace de Z à partir de conditions moins restrictives que précédemment.

THEOREM 2.2. – *Avec les notations du théorème 2.1, et sous la condition (1), lorsqu'il existe deux variable aléatoires $\mathcal{A}$-mesurables V et W telles que*

$$\sum_{k=1}^n Z - Z_k \leqslant V \ p.s. \ \ et \ \ \sum_{k=1}^n \mathbb{E}_n^k\left[(Z'_k)^2\right] \leqslant W \ p.s.\,,$$

alors, pour tout $\theta > 0$, et tout $\lambda \in [0, (1+u)/\theta)$,

$$\log \mathbb{E}\left[e^{\lambda(Z-\mathbb{E}[Z])}\right] \leqslant \frac{\lambda}{1 - \lambda\theta/(1+u)}\left(\log \mathbb{E}\left[e^{\lambda V}\right] + \frac{\theta}{1+u}\log \mathbb{E}\left[e^{\lambda W/\theta}\right]\right)\,.$$

Enfin nous appliquons le théorème 2.1 pour obtenir un contrôle des déviations au dessus de sa moyenne du supremum d'un processus empirique indexé par une classe de fonctions bornées ou simplement bornées à droite.

THEOREM 2.3. – *Supposons les X_i equidistribués selon P. Soit $\mathcal{F}$ un ensemble dénombrable de fonctions de $\mathcal{X}$ dans $\mathbb{R}$ de carré intégrable et d'espérance nulle sous P. Si $\sup_{f\in\mathcal{F}} \operatorname{ess\,sup} f \leqslant 1$ alors on définit*

$$Z = \sup_{f\in\mathcal{F}} \sum_{i=1}^n f(X_i)\,,$$

et si $\sup_{f\in\mathcal{F}} \|f\|_\infty \leqslant 1$ on définit

$$Z = \sup_{f\in\mathcal{F}} \left|\sum_{i=1}^n f(X_i)\right|\,.$$

Soit alors σ un réel tel que $\sigma^2 \geqslant \sup_{f\in\mathcal{F}} \mathrm{Var}\left[f(X_1)\right]$ presque sûrement, alors pour tout $x \geqslant 0$, on a

$$\mathbb{P}\left[Z \geqslant \mathbb{E}[Z] + x\right] \leqslant \exp\left(-vh\left(\frac{x}{v}\right)\right),$$

où $v = n\sigma^2 + 2\mathbb{E}[Z]$ et aussi

$$\mathbb{P}\left[Z \geqslant \mathbb{E}[Z] + \sqrt{2xv} + \frac{x}{3}\right] \leqslant e^{-x}.$$

1. Introduction

We consider a sequence of independent random variables $X_1, \ldots, X_n$ with values in some polish space $\mathcal{X}$ and distributed according to P. Let F be a P-measurable function from $\mathcal{X}^n$ to $\mathbb{R}$. We are interested in the conditions that the random variable $Z = F(X_1, \ldots, X_n)$ should satisfy in order to be concentrated around its expectation. We provide two theorems that give upper bounds on the Laplace transform of Z under general conditions. We prove that these conditions are satisfied in particular by suprema of empirical processes, i.e. $Z = \sup_{f\in\mathcal{F}} \sum_{i=1}^n f(X_i)$ with $\mathcal{F}$ a countable family of P-measurable functions. This allows us to get a Bennett inequality for such random variables which improves on Rio's version of Talagrand's inequality.

For all $k = 1, \ldots, n$, let $\mathcal{A}_k$ be the sigma field generated by $(X_1, \ldots, X_k)$ and let $\mathcal{A}_n^k$ be the sigma field generated by $(X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n)$. We denote by $\mathbb{E}_n^k[.]$ the expectation taken conditionally to $\mathcal{A}_n^k$. Let $h(x) = (1+x)\log(1+x) - x$, $\psi(x) = e^{-x} - 1 + x$ and $\phi(x) = 1 - (1+x)e^{-x}$.

2. Main Results

The first result can be considered as a generalization of a result of Boucheron, Lugosi and Massart [1] since it gives a Bennett type concentration inequality for Z under less restrictive conditions.

THEOREM 2.1. – *Let $(Z, Z_1', \ldots, Z_n')$ be a sequence of $\mathcal{A}$-measurable r.v. and let $(Z_k)_{k=1,\ldots,n}$ be a sequence of r.v. respectively $\mathcal{A}_n^k$-measurable. Assume that there exists $u > 0$ such that for all $k = 1, \ldots, n$ the following inequalities are satisfied*

$$Z_k' \leqslant Z - Z_k \leqslant 1 \ a.s.\,, \ \mathbb{E}_n^k\left[Z_k'\right] \geqslant 0 \ \text{ and } \ Z_k' \leqslant u \ a.s. \tag{2}$$

Let σ be a real satisfying $\sigma^2 \geqslant \frac{1}{n}\sum_{k=1}^n \mathbb{E}_n^k\left[(Z_k')^2\right]$ almost surely and let $v = (1+u)\mathbb{E}[Z] + n\sigma^2$. If the following condition holds

$$\sum_{k=1}^n Z - Z_k \leqslant Z \ a.s.\,, \tag{3}$$

we obtain, for all $\lambda \geqslant 0$,

$$\log \mathbb{E}\left[e^{\lambda(Z - \mathbb{E}[Z])}\right] \leqslant \psi(-\lambda)v\,,$$

which gives the following bounds for all $x > 0$,

$$\mathbb{P}\left[Z \geqslant \mathbb{E}[Z] + x\right] \leqslant \exp\left(-vh\left(\frac{x}{v}\right)\right) \ \text{ and } \ \mathbb{P}\left[Z \geqslant \mathbb{E}[Z] + \sqrt{2vx} + \frac{x}{3}\right] \leqslant e^{-x}.$$

The second result relaxes further the conditions on Z. This allows to obtain upper bounds on the Laplace transform of Z of Bernstein type, provided one controls the Laplace transform of two quantities: the first being the sum of first order finite differences and the second being the sum of the squares of these differences. This result can be considered as a refinement of one of the results in [2].

THEOREM 2.2. – *With the notations of Theorem 2.1, when the condition (2) is satisfied, denoting by V and W two $\mathcal{A}$-measurable random variables such that*

$$\sum_{k=1}^{n} Z - Z_k \leqslant V \ a.s. \ \text{and} \ \sum_{k=1}^{n} \mathbb{E}_n^k \left[(Z_k')^2 \right] \leqslant W \ a.s.\,,$$

then for all $\theta > 0$, and all $\lambda \in [0, (1+u)/\theta)$ we have

$$\log \mathbb{E}\left[e^{\lambda(Z - \mathbb{E}[Z])} \right] \leqslant \frac{\lambda}{1 - \lambda\theta/(1+u)} \left(\log \mathbb{E}\left[e^{\lambda V} \right] + \frac{\theta}{1+u} \log \mathbb{E}\left[e^{\lambda W/\theta} \right] \right) \,.$$

The next result is an application of Theorem 2.1 which gives a functional generalization of Bennett's inequality. More precisely, it gives a bound on the deviation above its mean of the supremum of an empirical process indexed by a class of upper bounded or bounded functions. The bound we obtain reduces to the classical Bennett's inequality for sums of i.i.d. random variables when the index set is a singleton.

THEOREM 2.3. – *Assume the X_i are identically distributed according to P. Let $\mathcal{F}$ be a countable set of functions from $\mathcal{X}$ to $\mathbb{R}$ and assume that all functions f in $\mathcal{F}$ are P-measurable, square-integrable and satisfy $\mathbb{E}[f] = 0$. If $\sup_{f \in \mathcal{F}} \operatorname{ess\,sup} f \leqslant 1$ then we denote*

$$Z = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(X_i)\,,$$

and if $\sup_{f \in \mathcal{F}} \|f\|_\infty \leqslant 1$ we denote

$$Z = \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} f(X_i) \right| \,.$$

Let σ be a positive real number such that $\sigma^2 \geqslant \sup_{f \in \mathcal{F}} \mathsf{Var}\left[f(X_1)\right]$ almost surely, then for all $x \geqslant 0$, we have

$$\mathbb{P}\left[Z \geqslant \mathbb{E}[Z] + x \right] \leqslant \exp\left(-vh\left(\frac{x}{v} \right) \right) \,,$$

with $v = n\sigma^2 + 2\mathbb{E}[Z]$ and also

$$\mathbb{P}\left[Z \geqslant \mathbb{E}[Z] + \sqrt{2xv} + \frac{x}{3} \right] \leqslant e^{-x} \,.$$

This result improves the main result in [6] where the exponential rate is $-\frac{x}{2}\log(1 + \frac{x}{v})$ in the first inequality and the factor of x in the second inequality is $1/2$ instead of $1/3$. It also provides a positive answer to the question raised in [4] about the possibility of obtaining a functional version of Bennett's inequality with optimal constants.

3. Sketch of the proofs

In order to derive concentration inequalities around the expectation of the random variables, we use the so-called *entropy method* introduced by Ledoux [3], and further refined by Massart [4] and Rio [5] among others. In particular, many applications of this method have been exposed in [1, 2].

This method consists in obtaining bounds on the logarithmic Laplace transform of a random function on a product space from bounds on the first-order finite differences of this function. Two main steps are necessary. The first one consists in using the so-called tensorization property of entropy which allows to decompose the entropy of a function of n independent random variables into a sum of entropies with respect to each individual random variable. The second step uses a variational principle for the entropy to bound the entropy with respect to one variable in terms of a first-order partial difference. The result is a differential inequality involving the Laplace transform of the random function. Once integrated, this gives an upper bound on the log-Laplace transform which can be turned into a deviation inequality via the classical Markov's inequality.

The main tool in the entropy method is the following inequality by Massart [4].

LEMMA 3.1. – *Let Z be a $\mathcal{A}$-measurable random variable, then we have for all λ, and for any sequence $(Z_k)_{1\leqslant k\leqslant n}$ of respectively $\mathcal{A}_n^k$-measurable random variables,*

$$\lambda\mathbb{E}\left[Ze^{\lambda Z}\right]-\mathbb{E}\left[e^{\lambda Z}\right]\log\mathbb{E}\left[e^{\lambda Z}\right]\leqslant\mathbb{E}\left[\sum_{k=1}^{n}\psi\left(\lambda\left(Z-Z_k\right)\right)e^{\lambda Z}\right]. \tag{4}$$

We will also use the following lemma (see e.g. [4]) as a decoupling device.

LEMMA 3.2. – *If V and Z are two $\mathcal{A}$-measurable random variables, we have for any λ and any $\theta>0$,*

$$\lambda\mathbb{E}\left[Ve^{\lambda Z}\right]\leqslant\lambda\theta\mathbb{E}\left[Ze^{\lambda Z}\right]-\theta\mathbb{E}\left[e^{\lambda Z}\right]\log\mathbb{E}\left[e^{\lambda Z}\right]+\theta\mathbb{E}\left[e^{\lambda Z}\right]\log\mathbb{E}\left[e^{\lambda V/\theta}\right].$$

Using the above two lemmas, we can prove that if V is a $\mathcal{A}$-measurable r.v. such that $\sum_{k=1}^n Z-Z_k\leqslant V$ then we have for all $\lambda>0$,

$$\sum_{k=1}^{n}\mathbb{E}\left[e^{\lambda Z}-e^{\lambda Z_k}\right]\leqslant\mathbb{E}\left[e^{\lambda Z}\right]\log\mathbb{E}\left[e^{\lambda V}\right]. \tag{5}$$

Proof of Theorem 2.1. – We essentially refine an argument introduced by Rio [6]. We start with Inequality (4) and upper bound the summands in the right hand side term. A careful analysis of the properties of the functions ϕ and ψ leads to the following upper bound

$$\psi(\lambda(Z-Z_k))e^{\lambda Z}\leqslant\frac{\phi(-\lambda)}{\psi(-\lambda)+\lambda/(1+u)}\left(e^{\lambda Z}-e^{\lambda Z_k}+\lambda e^{\lambda Z_k}\left((1+u)^{-1}(Z_k')^2-Z_k'\right)\right). \tag{6}$$

Now, because of conditions (2) we have for any $\mathcal{A}$-measurable random variable U

$$\mathbb{E}\left[e^{\lambda Z_k}U\right]\leqslant\mathbb{E}\left[e^{\lambda Z}\mathbb{E}_n^k\left[U\right]\right].$$

Using this inequality and the fact that $\mathbb{E}[Z_k']\geqslant 0$ and summing up (6) for $k=1,\ldots,n$ we obtain

$$\mathbb{E}\left[\sum_{k=1}^{n}\psi\left(\lambda\left(Z-Z_k\right)\right)e^{\lambda Z}\right]\leqslant\frac{\phi(-\lambda)}{\psi(-\lambda)+\lambda/(1+u)}\mathbb{E}\left[\sum_{k=1}^{n}\left(e^{\lambda Z}-e^{\lambda Z_k}\right)+\frac{n\sigma^2\lambda}{1+u}e^{\lambda Z}\right].$$

Using Inequality (5) and plugging the result in Inequality (4) we thus obtain a differential inequality that has to be satisfied by $F(\lambda)=\mathbb{E}\left[e^{\lambda Z}\right]$, the Laplace transform of Z.

Integrating this inequality gives the upper bound on the Laplace transform and standard calculus using Markov's inequality gives the deviation bounds. □

Proof of Theorem 2.2. – Using a similar reasoning as in the proof of Theorem 2.1 we obtain

$$\mathbb{E}\left[\sum_{k=1}^{n}\psi\left(\lambda\left(Z-Z_k\right)\right)e^{\lambda Z}\right]\leqslant\frac{\phi(-\lambda)}{\psi(-\lambda)+\lambda/(1+u)}\mathbb{E}\left[Ve^{\lambda Z}+\frac{\lambda}{1+u}We^{\lambda Z}\right].$$

Then we use Lemma 3.2 to decouple $\mathbb{E}\left[We^{\lambda Z}\right]$ and use proof techniques inspired by [2] to integrate the resulting differential inequality. In particular we use the properties of the log-Laplace transforms (convexity, 0 in 0) to obtain a simple upper bound. □

Proof of Theorem 2.3. – Theorem 2.3 is an easy consequence of Theorem 2.1. The proof simply consists in proving that the random variable Z satisfies conditions (2) and (3). For this purpose, considering the case where Z is defined with absolute values (the other case is treated in the same way), we define the following auxiliary random variables for all $k = 1, \ldots, n$

$$Z_k = \sup_{f \in \mathcal{F}} \left| \sum_{i \neq k} f(X_i) \right| \quad \text{and} \quad Z'_k = \left| \sum_{i=1}^n f_k(X_i) \right| - Z_k \,,$$

where f_k denotes the function for which the supremum is reached in Z_k (we use f_0 for the function in Z). We then get

$$Z'_k \leqslant Z - Z_k \leqslant \left| \sum_{i=1}^n f_0(X_i) \right| - \left| \sum_{i \neq k} f_0(X_i) \right| \leqslant |f_0(X_k)| \leqslant 1 \text{ a.s. } .$$

Moreover, we have

$$\mathbb{E}_n^k \left[Z'_k \right] \geqslant \left| \mathbb{E}_n^k \left[\sum_{i=1}^n f_k(X_i) \right] \right| - Z_k = 0 \,,$$

which concludes the proof of (2) with $u = 1$. Also,

$$(n-1)Z = \left| \sum_{k=1}^n \sum_{i \neq k} f_0(X_i) \right| \leqslant \sum_{k=1}^n \left| \sum_{i \neq k} f_0(X_i) \right| \leqslant \sum_{k=1}^n Z_k \,,$$

which gives (3), and finally, since

$$\sum_{k=1}^n \mathbb{E}_n^k \left[(Z'_k)^2 \right] \leqslant \sum_{k=1}^n \mathbb{E}_n^k \left[f_k^2(X_k) \right] \leqslant n \sup_{f \in \mathcal{F}} \mathsf{Var}\left[f(X_1) \right] \,,$$

we can choose σ as proposed. □

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Appendix : Detailed Proofs

Proof of Inequality (5). – We denote $F(\lambda) = \mathbb{E}\left[e^{\lambda Z}\right]$ and $G(\lambda) = \mathbb{E}\left[e^{\lambda V}\right]$. We apply Lemma 3.1 to the variable Z with the sequence of variables $Z_k + \frac{1}{n\lambda}\log G(\lambda)$. We obtain

$$\begin{aligned}
&\lambda F'(\lambda) - F(\lambda)\log F(\lambda) \\
&\leqslant \sum_{k=1}^{n} \mathbb{E}\left[G(\lambda)^{1/n} e^{\lambda Z_k} - e^{\lambda Z} + e^{\lambda Z}(\lambda(Z - Z_k) - \frac{1}{n}\log G(\lambda))\right] \\
&\leqslant \sum_{i=1}^{k} G(\lambda)^{1/n}\mathbb{E}\left[e^{\lambda Z_k}\right] - nF(\lambda) + \lambda\mathbb{E}\left[Ve^{\lambda Z}\right] - F(\lambda)\log G(\lambda) \\
&\leqslant \sum_{i=1}^{k} G(\lambda)^{1/n}\mathbb{E}\left[e^{\lambda Z_k}\right] - nF(\lambda) + \lambda F'(\lambda) - F(\lambda)\log F(\lambda)\,,
\end{aligned}$$

where we used Lemma 3.2 in the last step. This gives

$$G(\lambda)^{1/n}\sum_{k=1}^{n}\mathbb{E}\left[e^{\lambda Z_k}\right] \geqslant nF(\lambda)\,,$$

and thus

$$\sum_{k=1}^{n}\mathbb{E}\left[e^{\lambda Z} - e^{\lambda Z_k}\right] \leqslant nF(\lambda)(1 - G(\lambda)^{-1/n})\,.$$

Now, using the fact that $e^x \geqslant 1 + x$ we obtain the result. □

Proof of Theorem 2.1. – We define

$$f(\lambda) := \frac{\phi(-\lambda)}{\psi(-\lambda) + \lambda\alpha}\,.$$

We first prove the following inequality, which is valid for all $x \leqslant 1$, all $\lambda \geqslant 0$ and all $\alpha \geqslant 0$,

$$\psi(\lambda x) \leqslant f(\lambda)\left(\phi(\lambda x) + \lambda\alpha x^2 e^{-\lambda x}\right)\,. \tag{7}$$

Denote by h the function defined as

$$h(x) := \psi(\lambda x) - f(\lambda)\left(\phi(\lambda x) + \lambda\alpha x^2 e^{-\lambda x}\right)\,.$$

Without loss of generality, we consider the case $\lambda > 0$. We have $h(0) = h(1) = 0$ and $h'(0) = 0$. Moreover, we can see that $\lim_{x\to+\infty} h'(x) = \lambda$ and $\lim_{x\to-\infty} h'(x) = +\infty$. We can check that $h''(x) = e^{-\lambda x}p(x)$ where p is a second degree polynomial whose leading term is $-\lambda^3 f(\lambda)\alpha$. Thus $h''(x) = 0$ has at most two solutions. If there is no solution, then h' is non-increasing which contradicts the above facts about h'. Let's thus denote by x_1 and x_2 the two solutions (with $x_1 \leqslant x_2$ and possibly $x_1 = x_2$). h' is thus decreasing for $x < x_1$ and $x > x_2$ and it is increasing for $x \in (x_1, x_2)$. Because $\lim_{x\to-\infty} h'(x) = +\infty$, the equation $h'(x) = 0$ can have at most two solutions, one of which is 0. Let's denote by x_3 the other solution, we can have one of the following situations :

1. $x_3 = 0$. In that case, h is increasing for $x \neq 0$ which contradicts the fact that $h(0) = h(1) = 0$ and $\lambda > 0$.

2. $x_3 < 0$. This means that h is increasing for $x > 0$ which again leads to a contradiction.
3. $x_3 > 0$. In that case h is increasing for $x < 0$, decreasing for $x \in (0, x_3)$ and increasing for $x > x_3$.

Since $h(0) = h(1) = 0$ this proves that $h(x) \leqslant 0$ for $x \leqslant 1$ which is what we needed.

We thus have proved (7) and thus using $x = Z - Z_k$ we get

$$\begin{aligned}\psi(\lambda(Z - Z_k))e^{\lambda Z} &\leqslant f(\lambda)(\psi(-\lambda(Z - Z_k)) + \lambda\alpha(Z - Z_k)^2)e^{\lambda Z_k} \\ &= f(\lambda)\left(e^{\lambda Z} - e^{\lambda Z_k}\right) + \lambda f(\lambda)e^{\lambda Z_k}\left(\alpha(Z - Z_k)^2 - (Z - Z_k)\right)\end{aligned}$$

Now, we choose $\alpha = 1/(1+u)$ so that when $y \leqslant x \leqslant 1$ and $y \leqslant u$ we have $\alpha x^2 - x \leqslant \alpha y^2 - y$. Using the fact that $Z_k' \leqslant Z - Z_k \leqslant 1$ and $Z_k' \leqslant u$, this allows to upper bound the above as

$$\psi(\lambda(Z - Z_k))e^{\lambda Z} \leqslant f(\lambda)\left(e^{\lambda Z} - e^{\lambda Z_k}\right) + \lambda f(\lambda)e^{\lambda Z_k}\left((1+u)^{-1}(Z_k')^2 - Z_k'\right).$$

Now taking the expectation and using $\mathbb{E}_n^k\left[Z_k'\right] \geqslant 0$, we obtain

$$\begin{aligned}\mathbb{E}\left[\psi(\lambda(Z - Z_k))e^{\lambda Z}\right] &\leqslant f(\lambda)\mathbb{E}\left[e^{\lambda Z} - e^{\lambda Z_k}\right] + \frac{\lambda f(\lambda)}{1+u}\mathbb{E}\left[e^{\lambda Z_k}\mathbb{E}_n^k\left[(Z_k')^2\right]\right] \\ &\leqslant f(\lambda)\mathbb{E}\left[e^{\lambda Z} - e^{\lambda Z_k}\right] + \frac{\lambda f(\lambda)}{1+u}\mathbb{E}\left[e^{\lambda Z}\mathbb{E}_n^k\left[(Z_k')^2\right]\right],\end{aligned}$$

where we used $\mathbb{E}_n^k\left[e^{\lambda Z_k}\right] \leqslant \mathbb{E}_n^k\left[e^{\lambda Z}\right]$ (which follows from using $\mathbb{E}_n^k\left[Z\right] \geqslant Z_k$ and Jensen's inequality).

Now let's denote $F(\lambda) = \mathbb{E}\left[e^{\lambda Z}\right]$. Summing the above inequality for $k = 1, \ldots, n$ and using Lemma 3.1 we thus obtain the following differential inequality, valid for all $\lambda \geqslant 0$

$$\lambda F'(\lambda) - F(\lambda)\log F(\lambda) \leqslant f(\lambda)\mathbb{E}\left[\sum_{k=1}^n e^{\lambda Z} - e^{\lambda Z_k}\right] + \frac{n\sigma^2\lambda f(\lambda)}{1+u}F(\lambda).$$

Using the fact that $\sum_{k=1}^n Z - Z_k \leqslant Z$ with Inequality (5) we get

$$\lambda F'(\lambda) - F(\lambda)\log F(\lambda) \leqslant f(\lambda)F(\lambda)\log F(\lambda) + \frac{n\sigma^2\lambda f(\lambda)}{1+u}F(\lambda).$$

Let $G(\lambda) = \mathbb{E}\left[e^{\lambda(Z - \mathbb{E}[Z])}\right]$, we get

$$\lambda\frac{G'(\lambda)}{G(\lambda)} - \log G(\lambda) \leqslant f(\lambda)(\log G(\lambda) + \lambda\mathbb{E}[Z]) + \frac{n\sigma^2\lambda}{1+u}f(\lambda).$$

We can rewrite this inequality as

$$\lambda\frac{G'(\lambda)}{G(\lambda)} - (1 + f(\lambda))\log G(\lambda) \leqslant \frac{(n\sigma^2 + (1+u)\mathbb{E}[Z])\lambda}{1+u}f(\lambda).$$

Denoting $L(\lambda) = \log G(\lambda)$, we obtain

$$\lambda L'(\lambda) - (1 + f(\lambda))L(\lambda) \leqslant \frac{(n\sigma^2 + (1+u)\mathbb{E}[Z])\lambda}{1+u}f(\lambda). \tag{8}$$

We easily check that $L_0(\lambda) := (n\sigma^2 + (1+u)\mathbb{E}[Z])\psi(-\lambda)$ is a solution of the associated differential equation with $L_0(0) = 0$ and $L_0'(0) = 0$. Thus, for any solution L of (8) with $L(0) = L'(0) = 0$, we have for $\lambda \geqslant 0$

$$L(\lambda) \leqslant L_0(\lambda).$$

We thus have

$$\log\mathbb{E}\left[e^{\lambda(Z - \mathbb{E}[Z])}\right] \leqslant (n\sigma^2 + (1+u)\mathbb{E}[Z])\psi(-\lambda),$$

which proves the first inequality. The deviation bounds follows from standard calculations. □

Proof of Theorem 2.2. – As in the proof of Theorem 2.1 we define We define

$$f(\lambda) := \frac{\phi(-\lambda)}{\psi(-\lambda) + \lambda\alpha} ,$$

with $\alpha = 1/(1+u)$.

Now notice that for all $\lambda \geqslant 0$ and $u \in [0,1]$, we have

$$(1 - \frac{1}{1+u})\lambda^2 \leqslant \frac{\lambda^2}{2} \leqslant e^\lambda - 1 - \lambda ,$$

so that we obtain after simple algebra

$$f(\lambda) \leqslant \lambda .$$

We proceed as in the proof of Theorem 2.1 to get

$$\mathbb{E}\left[\psi(\lambda(Z - Z_k))e^{\lambda Z}\right] \leqslant f(\lambda)\mathbb{E}\left[e^{\lambda Z} - e^{\lambda Z_k}\right] + \frac{\lambda f(\lambda)}{1+u}\mathbb{E}\left[e^{\lambda Z}\mathbb{E}_n^k\left[(Z_k')^2\right]\right] ,$$

which by summing over $k = 1, \ldots, n$ and using the notation $F(\lambda) = \mathbb{E}\left[e^{\lambda Z}\right]$ gives

$$\lambda F'(\lambda) - F(\lambda)\log F(\lambda) \leqslant f(\lambda)\mathbb{E}\left[\sum_{k=1}^n e^{\lambda Z} - e^{\lambda Z_k}\right] + \frac{\lambda f(\lambda)}{1+u}\mathbb{E}\left[We^{\lambda Z}\right] .$$

Using Inequality (5) and Lemma 3.2 we obtain

$$\left(1 - \frac{f(\lambda)\theta}{1+u}\right)(\lambda F'(\lambda) - F(\lambda)\log F(\lambda)) \leqslant f(\lambda)F(\lambda)\left(\log \mathbb{E}\left[e^{\lambda V}\right] + \frac{\theta}{1+u}\log \mathbb{E}\left[e^{\lambda W/\theta}\right]\right) ,$$

for all $\theta \geqslant 0$.

Since without loss of generality we can assume that $F(\lambda) > 0$, we can rewrite the above for $0 < f(\lambda) < (1+u)/\theta$ as

$$\frac{1}{\lambda}\frac{F'(\lambda)}{F(\lambda)} - \frac{1}{\lambda^2}\log F(\lambda) \leqslant \frac{f(\lambda)}{\lambda^2(1 - f(\lambda)\theta/(1+u))}\left(G(\lambda) + \frac{\theta}{1+u}H(\lambda/\theta)\right) ,$$

where $G(\lambda) = \log \mathbb{E}\left[e^{\lambda V}\right]$ and $H(\lambda) = \log \mathbb{E}\left[e^{\lambda W}\right]$.

Since $f(\lambda) \leqslant \lambda$, the condition $0 < f(\lambda) < (1+u)/\theta$ is satisfied when $0 < \lambda < (1+u)/\theta$. For any such λ, we can integrate the above differential inequality since the left hand side is simply the derivative of $\lambda \mapsto \frac{1}{\lambda}\log F(\lambda)$ (which is equal to $\mathbb{E}[Z]$ for $\lambda = 0$). We thus get

$$\frac{1}{\lambda}\log F(\lambda) - \mathbb{E}[Z] \leqslant \int_0^\lambda \frac{f(s)}{s^2(1 - f(s)\theta/(1+u))}\left(G(s) + \frac{\theta}{1+u}H(s/\theta)\right) ds ,$$

Now notice that since G is convex and $G(0) = 0$, $G(\lambda)/\lambda$ is non-decreasing for $\lambda > 0$. The same is true for H. We can thus upper bound the right hand side by

$$\frac{1}{\lambda}\left(G(\lambda) + \frac{\theta}{1+u}H(\lambda/\theta)\right)\int_0^\lambda \frac{f(s)}{s(1 - f(s)\theta/(1+u))}ds .$$

Now using $f(s) \leqslant s$ we get

$$\int_0^\lambda \frac{f(s)}{s(1 - f(s)\theta/(1+u))}ds \leqslant \int_0^\lambda \frac{1}{(1 - s\theta/(1+u))}ds \leqslant \frac{\lambda}{1 - \lambda\theta/(1+u)} ,$$

where we used $\log x \leqslant x - 1$.

Finally we obtain

$$\log \mathbb{E}\left[e^{\lambda(Z-\mathbb{E}[Z])}\right] \leqslant \frac{\lambda}{1-\lambda\theta/(1+u)}\left(G(\lambda)+\frac{\theta}{1+u}H(\lambda/\theta)\right),$$

which completes the proof. $\square$

Proof of Theorem 2.3. – We first consider the case $\operatorname{ess\,sup} f \leqslant 1$.

We denote by f_k a function such that

$$\sum_{i\neq k} f_k(X_i) = \sup_{f\in\mathcal{F}} \sum_{i\neq k} f(X_i)\,.$$

This function always exists when $\mathcal{F}$ is finite. We will thus restrict to this case. The result for a countable $\mathcal{F}$ is easily derived by taking the limit of a sequence of finite sets.

We introduce the following auxiliary random variables for $k = 1, \ldots, n$,

$$Z_k = \sup_{f\in\mathcal{F}} \sum_{i\neq k} f(X_i) \text{ and } Z'_k = f_k(X_k)\,,$$

Denoting by f_0 the function achieving the maximum in Z, we have

$$Z - Z_k \leqslant f_0(X_k) \leqslant 1 \text{ a.s. },$$

$$Z - Z_k - Z'_k \geqslant 0\,,$$

and since f_k is $\mathcal{F}_n^k$-measurable, we have

$$\mathbb{E}_n^k\left[Z'_k\right] = 0\,.$$

Also we have

$$(n-1)Z = \sum_{k=1}^n \sum_{i\neq k} f_0(X_i) = \sum_{k=1}^n \sum_{i\neq k} f_0(X_i) \leqslant \sum_{k=1}^n Z_k\,,$$

and

$$\sum_{k=1}^n \mathbb{E}_n^k\left[(Z'_k)^2\right] = \sum_{k=1}^n \mathbb{E}_n^k\left[f_k^2(X_k)\right] \leqslant \sum_{k=1}^n \sup_{f\in\mathcal{F}} \mathrm{Var}\left[f(X_k)\right] = n \sup_{f\in\mathcal{F}} \mathrm{Var}\left[f(X_1)\right]\,.$$

Notice that we used the fact that the X_i have identical distribution. We refer to the remark in [6] for the general case.

Now, in the case $\|f\|_\infty \leqslant 1$ the reasoning is similar. We introduce for $k = 1, \ldots, n$,

$$Z_k = \sup_{f\in\mathcal{F}} \left|\sum_{i\neq k} f(X_i)\right| \text{ and } Z'_k = \left|\sum_{i=1}^n f_k(X_i)\right| - Z_k\,.$$

We have

$$Z'_k \leqslant Z - Z_k \leqslant \left|\sum_{i=1}^n f_0(X_i)\right| - \left|\sum_{i\neq k} f_0(X_i)\right| \leqslant |f_0(X_k)| \leqslant 1 \text{ a.s. }.$$

Moreover, we have

$$\mathbb{E}_n^k\left[Z'_k\right] \geqslant \left|\mathbb{E}_n^k\left[\sum_{i=1}^n f_k(X_i)\right]\right| - Z_k = 0\,,$$

which concludes the proof of (2) with $u = 1$. Also,

$$(n-1)Z = \left| \sum_{k=1}^{n} \sum_{i \neq k} f_0(X_i) \right| \leqslant \sum_{k=1}^{n} \left| \sum_{i \neq k} f_0(X_i) \right| \leqslant \sum_{k=1}^{n} Z_k \,,$$

which gives (3), and finally, since

$$\sum_{k=1}^{n} \mathbb{E}_n^k \left[(Z_k')^2 \right] \leqslant \sum_{k=1}^{n} \mathbb{E}_n^k \left[f_k^2(X_k) \right] \leqslant \sum_{k=1}^{n} \sup_{f \in \mathcal{F}} \mathsf{Var}\left[f(X_k) \right] = n \sup_{f \in \mathcal{F}} \mathsf{Var}\left[f(X_1) \right] \,,$$

we can choose σ as proposed.

□